On a subclass of α-uniform convex functions

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Abstract

In this paper we define a subclass of α-uniform convex functions by using the Sălăgean differential operator and we obtain some properties of this class.

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1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) \;:\; f\, is\, univalent\, in\, U\}$ and $S = \{f \in A : f\, is\, univalent\, in\, U\}$.

Let consider the integral operator $L_a : A \to A$ defined as:

(1) $$f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt, \quad a \in \mathbb{C},\; Re\; a \geq 0.$$

In the case $a = 1, 2, 3, ...$ this operator was introduced by S.D. Bernardi and it was studied by many authors in different general cases. In the form (1) was used first time by N.N. Pascu.

Let D^n be the Sălăgean differential operator (see [8]) defined as:

$$D^n : A \to A\,,\; n \in \mathbb{N}\; and\; D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)\;,\; D^n f(z) = D\left(D^{n-1} f(z)\right)$$

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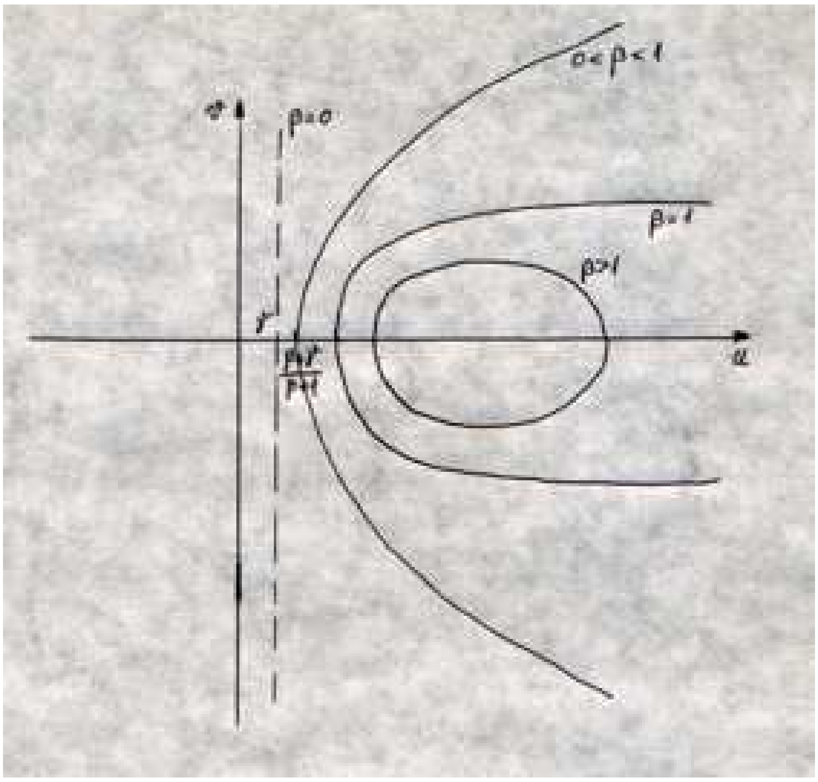
2 Preliminary results

Definition 2.1 *[3] Let $f \in A$. We say that f is n-uniform starlike function of order γ and type β if*

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \geq \beta \cdot \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| + \gamma \quad , z \in U$$

where $\beta \geq 0$, $\gamma \in [-1, 1)$, $\beta + \gamma \geq 0$, $n \in \mathbb{N}$. We denote this class with $US_n(\beta, \gamma)$.

Remark 2.1 *Geometric interpretation: $f \in US_n(\beta, \gamma)$ if and only if $\frac{D^{n+1}f(z)}{D^n f(z)}$ take all values in the convex domain included in right half plane $D_{\beta,\gamma}$, where $D_{\beta,\gamma}$ is a eliptic region for $\beta > 1$, a parabolic region for $\beta = 1$, a hiperbolic region for $0 < \beta < 1$, the half plane $u > \gamma$ for $\beta = 0$.*



Remark 2.2 *If we take $n = 0$ and $\beta = 1$ in definition 2.1 we obtain $US_0(1, \gamma) = SP\left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$, where the class $SP(\alpha, \beta)$ was introduced by F. Ronning in [9]. Also we have $US_n(\beta, \gamma) \subset S^*$, where S^* is the well know class of starlike functions.*

Definition 2.2 *[3] Let $f \in A$. We say that f is α-uniform convex function, $\alpha \in [0, 1]$ if*

$$Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \geq \left|(1-\alpha)\left(\frac{zf'(z)}{f(z)} - 1\right) + \alpha\frac{zf''(z)}{f'(z)}\right|, \ z \in U.$$

We denote this class with UM_α.

Remark 2.3 *Geometric interpretation:* $f \in UM_\alpha$ *if and only if*

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

take all values in the parabolic region $\Omega = \{w : |w-1| \leq Rew\} = \{w = u+iv\, ;\, v^2 \leq 2u-1\}$. *Also we have* $UM_\alpha \subset M_\alpha$, *where* M_α *is the well know class of* α*-convex functions introduced by P.T. Mocanu in [7].*

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [4],[5],[6]).

Theorem 2.1 *Let* h *convex in* $\mathcal{U}$ *and* $Re[\beta h(z) + \delta] > 0,\ z \in \mathcal{U}$. *If* $p \in \mathcal{H}(\mathcal{U})$ *with* $p(0) = h(0)$ *and* p *satisfied the Briot-Bouquet differential subordination* $p(z) + \dfrac{zp'(z)}{\beta p(z) + \delta} \prec h(z)$, *then* $p(z) \prec h(z)$.

3 Main results

Definition 3.1 *Let* $\alpha \in [0,1]$ *and* $n \in \mathbb{N}$. *We say that* $f \in A$ *is in the class* $UD_{n,\alpha}(\beta,\gamma)$, $\beta \geq 0$, $\gamma \in [-1,1)$, $\beta + \gamma \geq 0$ *if*

$$Re\left[(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right] \geq \beta\left|(1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right| + \gamma$$

Remark 3.1 *We have* $UD_{n,0}(\beta,\gamma) = US_n(\beta,\gamma) \subset S^*$,$UD_{0,\alpha}(1,0) = UM_\alpha$ *and* $UD_{0,1}(\beta,\gamma) = US^c(\beta,\gamma) \subset S^c\left(\dfrac{\beta+\gamma}{\beta+1}\right)$ *where* $US^c(\beta,\gamma)$ *is the class of the uniform convex functions of type* β *and order* γ *introduced by I. Magdaş in [2] and* $S^c(\delta)$ *is the well know class of convex functions of order* δ.

Remark 3.2 *Geometric interpretation:* $f \in UD_{n,\alpha}(\beta,\gamma)$ *if and only if*

$$J_n(\alpha, f; z) = (1-\alpha)\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha\frac{D^{n+2}f(z)}{D^{n+1}f(z)}$$

take all values in the convex domain included in right half plane $D_{\beta,\gamma}$, *where* $D_{\beta,\gamma}$ *is defined in remark 2.1.*

Theorem 3.1 *For all* $\alpha, \alpha' \in [0,1]$ *with* $\alpha < \alpha'$ *we have* $UD_{n,\alpha'}(\beta,\gamma) \subset UD_{n,\alpha}(\beta,\gamma)$.

Proof.

From $f \in UD_{n,\alpha'}(\beta,\gamma)$ we have

$$Re\left[(1-\alpha')\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha'\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right] \geq \beta\left|(1-\alpha')\frac{D^{n+1}f(z)}{D^n f(z)} + \alpha'\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right| + \gamma$$

With notations $\dfrac{D^{n+1}f(z)}{D^n f(z)} = p(z)$,where $p(z) = 1 + p_1 z + \cdots$ we have

$$zp'(z) = z\frac{(D^{n+1}f(z))' \cdot D^n f(z) - D^{n+1}f(z) \cdot (D^n f(z))'}{(D^n f(z))^2} =$$

$$= \frac{D^{n+2}f(z)}{D^n f(z)} - \left(\frac{D^{n+1}f(z)}{D^n f(z)}\right)^2$$

$$\frac{zp'(z)}{p(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^n f(z)}$$

and thus we obtain

$$J_n(\alpha', f; z) = p(z) + \alpha' \cdot \frac{zp'(z)}{p(z)}$$

Now we have $p(z) + \alpha' \cdot \dfrac{zp'(z)}{p(z)}$ take all values in the convex domain included in right half plane $D_{\beta,\gamma}$.

If we consider $h \in \mathcal{H}_u(U)$, with $h(0) = 1$, which maps the unit disc U into the convex domain $D_{\beta,\gamma}$, we have $Reh(z) > 0$ and from hypothesis $\alpha' > 0$. From here follows that $Re\dfrac{1}{\alpha'} \cdot h(z) > 0$. In this conditions from theorem 2.1 , whit $\delta = 0$ we obtain $p(z) \prec h(z)$, or $p(z)$ take all values in $D_{\beta,\gamma}$.

If we consider the function $g : [0, \alpha'] \to \mathbb{C}$, $g(u) = p(z) + u \cdot \dfrac{zp'(z)}{p(z)}$, with $g(0) = p(z) \in D_{\beta,\gamma}$ and $g(\alpha') \in D_{\beta,\gamma}$. Since the geometric image of $g(\alpha)$ is on the segment obtained by the union of the geometric image of $g(0)$ and $g(\alpha')$, we have $g(\alpha) \in D_{\beta,\gamma}$, or

$$p(z) + \alpha \cdot \frac{zp'(z)}{p(z)} \in D_{\beta,\gamma}$$

. Thus $J_n(\alpha, f; z)$ take all values in $D_{\beta,\gamma}$, or $f \in UD_{n,\alpha}(\beta, \gamma)$.

Remark 3.3 *From theorem 3.1 we have* $UD_{n,\alpha}(\beta,\gamma) \subset UD_{n,0}(\beta,\gamma)$ *for all* $\alpha \in [0, 1]$, *and from remark 3.1 we obtain that the functions from the class* $UD_{n,\alpha}(\beta, \gamma)$ *are univalent.*

Theorem 3.2 *If* $F(z) \in UD_{n,\alpha}(\beta, \gamma)$ *then* $f(z) = L_a(F)(z) \in US_n(\beta, \gamma)$, *where* L_a *is the integral operator defined by (1).*

Proof.

From (1) we have

$$(1 + a)F(z) = af(z) + zf'(z)$$

By means of the application of the linear operator D^{n+1} we obtain

$$(1 + a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+1}(zf'(z))$$

or

$$(1 + a)D^{n+1}F(z) = aD^{n+1}f(z) + D^{n+2}f(z)$$

Thus:

$$\frac{D^{n+1}F(z)}{D^nF(z)} = \frac{D^{n+2}f(z) + aD^{n+1}f(z)}{D^{n+1}f(z) + aD^nf(z)} =$$

$$= \frac{\dfrac{D^{n+2}f(z)}{D^{n+1}f(z)} \cdot \dfrac{D^{n+1}f(z)}{D^nf(z)} + a \cdot \dfrac{D^{n+1}f(z)}{D^nf(z)}}{\dfrac{D^{n+1}f(z)}{D^nf(z)} + a}$$

With notation $\dfrac{D^{n+1}f(z)}{D^nf(z)} = p(z)$ where $p(z) = 1 + p_1z + ...$ we have:

$$zp'(z) = z \cdot \left(\frac{D^{n+1}f(z)}{D^nf(z)}\right)' = \frac{z\left(D^{n+1}f(z)\right)' \cdot D^nf(z) - D^{n+1}f(z) \cdot z\left(D^nf(z)\right)'}{\left(D^nf(z)\right)^2} =$$

$$= \frac{D^{n+2}f(z) \cdot D^nf(z) - \left(D^{n+1}f(z)\right)^2}{\left(D^nf(z)\right)^2}$$

and

$$\frac{1}{p(z)} \cdot zp'(z) = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - \frac{D^{n+1}f(z)}{D^nf(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - p(z)$$

It follows:

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = p(z) + \frac{1}{p(z)} \cdot zp'(z)$$

Thus we obtain:

$$\frac{D^{n+1}F(z)}{D^nF(z)} = \frac{p(z) \cdot \left(zp'(z) \cdot \frac{1}{p(z)} + p(z)\right) + a \cdot p(z)}{p(z) + a} =$$

$$= p(z) + \frac{1}{p(z) + a} \cdot zp'(z)$$

If we denote $\dfrac{D^{n+1}F(z)}{D^nF(z)} = q(z)$, with $q(0) = 1$, and we consider $h \in \mathcal{H}_u(U)$, with $h(0) = 1$, which maps the unit disc U into the convex domain included in right half plane $D_{\beta,\gamma}$, we have from $F(z) \in UD_{n,\alpha}(\beta,\gamma)$ (see remark 3.2):

$$q(z) + \alpha \cdot \frac{zq'(z)}{q(z)} \prec h(z)$$

From theorem 2.1 , whit $\delta = 0$ we obtain $q(z) \prec h(z)$, or

$$p(z) + \frac{1}{p(z) + a} \cdot zp'(z) \prec h(z)$$

Using the hypothesis and the construction of the function $h(z)$ we obtain from theorem 2.1 $p(z) \prec h(z)$ or $f(z) \in US_n(\beta,\gamma)$ (see remark 2.1).

Remark 3.4 *From theorem 3.2 with $\alpha = 0$ we obtain the theorem 3.1 from [1] which assert that the integral operator L_a, defined by (1), preserve the class $US_n(\beta,\gamma)$.*

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