Some integral operators wich preserve a subclass of uniform quasiconvex functions

Mugur Acu*

Abstract

In this paper we define a subclass of uniform quasiconvex functions and we show that this class is preserve by the Alexander integral operator and the Bernardi integral operator.

2000 Mathematics Subject Classifications: 30C45

Key words and Phrases: Alexander integral operator, Bernardi integral operator, uniform quasiconvex functions, Ruscheweyh operator

1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, and $S = \{f \in A : f\, is\, univalent\, in\, U\}$.

Let denote by I the Alexander integral operator $I : A \to A$,

$$F(z) = If(z) = \int_0^z \frac{f(t)}{t} dt \tag{1}$$

,and by I_c the Bernardi integral operator $I_c : A \to A$,

$$F(z) = I_c f(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt \quad , c = 1, 2, 3, ... \tag{2}$$

*"Lucian Blaga" University, Department of Mathematics, 2400 Sibiu, Romania

2 Preliminary results

We denote by R^n the Ruscheweyh operator (see [10]) defined as:

$$R^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!} \quad , z \in U\,,\, n \in \mathbb{N}$$

where * is the convolution product.

Remark 2.1 *If $h \in A$, $h(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $z \in U$, then*

$$R^n h(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j$$

Let D^n be the Sălăgean differential operator (see [8]) defined as:

$$D^n : A \to A\,,\, n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)\ ,\ D^n f(z) = D\left(D^{n-1} f(z)\right)$$

Definition 2.1 *[4] Let $n \in \mathbb{N}$ and $f \in A$. We say that f is the class $UK_n(\delta)$, $\delta \in [-1,1)$, if*

$$Re\left(\frac{R^{n+1} f(z)}{R^n f(z)}\right) \geq \left|\frac{R^{n+1} f(z)}{R^n f(z)} - 1\right| + \delta \quad , z \in U$$

Remark 2.2 *Geometric interpretation: $f \in UK_n(\delta)$ if and only if $\frac{R^{n+1} f(z)}{R^n f(z)}$ take all values in the domain included in right half plane Ω_δ which is bounded by the parabola $v^2 = 2(1-\delta)u - (1-\delta^2)$. The Carathéodory functions is*

$$Q_\delta(z) = 1 + \frac{2(1-\delta)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \quad , z \in U \tag{3}$$

Thus, $f \in UK_n(\delta)$ if and only if $\frac{R^{n+1} f(z)}{R^n f(z)} \prec Q_\delta$, where by $\prec$ we denote the relation of subordination.

More, we know that the function Q_δ is convex and $Re\, Q_\delta > \frac{1+\delta}{2}$.

Remark 2.3 *If we take $n = 0$ in definition 2.1 we obtain $UK_0(\delta) = SP\left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right)$, where $SP(\alpha,\beta)$, $\alpha > 0$, $\beta \in [0,1)$ is the class of function $f \in S$ which satisfies the condition:*

$$\left|\frac{zf'(z)}{f(z)} - (\alpha+\beta)\right] \leq Re\frac{zf'(z)}{f(z)} + \alpha - \beta \quad , z \in U\,.$$

The class $SP(\alpha,\beta)$ was introduced by Ronning in [9].

Remark 2.4 *If we take $n = 1$ and $\delta = \frac{1}{2}$ in definition 2.1 we obtain $UK_1\left(\frac{1}{2}\right) = US^c$, where US^c is the class of uniform convex functions introduced by Goodman in [3].*

Definition 2.2 *[4] Let $f \in A$. We say that f is n-uniform starlike function of order δ and type α if*

$$Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| + \delta \quad , z \in U$$

where $\alpha \geq 0$, $\delta \in [-1,1)$, $\alpha + \delta \geq 0$, $n \in \mathbb{N}$. We denote this class with $US_n(\alpha, \delta)$.

Definition 2.3 *[2] Let $f \in A$. We say that f is n-uniform close to convex function of order δ and type α in respect to the function n-uniform starlike of order δ and type α $g(z)$, where $\alpha \geq 0$, $\delta \in [-1,1)$, $\alpha + \delta \geq 0$, if*

$$Re\left(\frac{D^{n+1}f(z)}{D^n g(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right| + \delta \quad , z \in U$$

where $\alpha \geq 0$, $\delta \in [-1,1)$, $\alpha + \delta \geq 0$, $n \in \mathbb{N}$. We denote this class with $UCC_n(\alpha, \delta)$.

Remark 2.5 *We have $UCC_n(\alpha, \delta) \subset CC$, where CC is the class of the close to convex functions defined by Kaplan which are univalent.*

Remark 2.6 *If we take $n = 0$ and $\alpha = 1$ in definition 2.2 we obtain $US_0(1, \delta) = SP\left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right)$.*

Theorem 2.1 *[4] If $f(z) \in UK_n(\delta)$, with $n \in \mathbb{N}$, $\delta \in [-1,1)$ and $c \in \mathbb{C}$ wtih $Re\, c \geq \frac{n(1-\delta)-(1+\delta)}{2}$, then $F(z) = I_c f(z) \in UK_n(\delta)$, where I_c is the Bernardi integral operator defined in (2).*

Theorem 2.2 *[1] If $f(z) \in UK_n(\delta)$, with $n \in \mathbb{N}$, $\delta \in [-1,1)$ and $\delta \geq \frac{n-1}{n+1}$ then $F(z) = If(z) \in UK_n(\delta)$, where I is the Alexander integral operator defined in (1).*

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [5],[6],[7]).

Theorem 2.3 *Let q be convex in U and $j : U \to \mathbb{C}$ with $Re[j(z)] > 0$, $z \in U$. If $p \in \mathcal{H}(U)$ and satisfied $p(z) + j(z) \cdot zp'(z) \prec q(z)$, then $p(z) \prec q(z)$.*

3 Main results

Definition 3.1 *Let $n \in \mathbb{N}$ and $f \in A$. We say that f is in the class $UQ_n(\delta)$, $\delta \in [-1,1)$, in respect to the function $g(z) \in UK_n(\delta)$, $\delta \in [-1,1)$ if*

$$Re\left(\frac{R^{n+1}f(z)}{R^n g(z)}\right) \geq \left|\frac{R^{n+1}f(z)}{R^n g(z)} - 1\right| + \delta \quad , z \in U$$

Remark 3.1 *Geometric interpretation:* $f \in UQ_n(\delta)$ *if and only if* $\dfrac{R^{n+1}f(z)}{R^n g(z)}$ *, where* $g(z) \in UK_n(\delta)$*, take all values in the domain* Ω_δ *which is bounded by the parabola* $v^2 = 2(1-\delta)u - (1-\delta^2)$*. The Carathédory function* Q_δ*, defined in (3), is convex and* $Re\, Q_\delta > \frac{1+\delta}{2}$*. Thus,* $f \in UQ_n(\delta)$ *in respect to the function* $g \in UK_n(\delta)$ *if and only if* $\dfrac{R^{n+1}f(z)}{R^n g(z)} \prec Q_\delta(z)$*.*

Remark 3.2 *If we take* $n = 0$ *in definition 3.1 we obtain that the subclass* $UQ_0(\delta)$ *,* $\delta \in [-1, 1)$*, is the class of the functions* $f \in A$ *which satisfies:*

$$Re \frac{zf'(z)}{g(z)} \geq \left| \frac{zf'(z)}{g(z)} - 1 \right| + \delta \ , \ z \in U$$

where $g \in UK_0(\delta) = US_0(1, \delta) = SP\left(\frac{1-\delta}{2}, \frac{1+\delta}{2}\right)$ *(see remarks 2.3 and 2.6). But, this class is the class* $UCC_0(1, \delta)$ *and thus from remark 2.5 we have that the functions from* $UQ_0(\delta)$ *are univalent.*

Remark 3.3 *Is easy to see that the function* $id(z) = z$ *,* $z \in U$ *satisfies* $id(z) \in UK_n(\delta)$ *for all* $n \in \mathbb{N}$ *and* $\delta \in [-1, 1)$*. It follows that* $id(z) \in UQ_n(\delta)$*, in respect to the function* $id(z) \in UK_n(\delta)$*, for all* $n \in \mathbb{N}$ *and* $\delta \in [-1, 1)$*.*

Theorem 3.1 *If* $f(z) \in UQ_n(\delta)$ *in respect to the function* $g(z) \in UK_n(\delta)$*, with* $n \in \mathbb{N}$*,* $\delta \in [-1, 1)$ *and* $\delta \geq \frac{n-1}{n+1}$*, then* $F(z) = If(z) \in UQ_n(\delta)$ *in respect to the function* $G(z) = Ig(z) \in UK_n(\delta)$*, where* I *is the Alexander integral operator defined by (1)*

Proof.

From theorem 2.2 we have $G(z) = Ig(z) \in UK_n(\delta)$ in the conditions from the hypothesis.

If we consider $w(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $z \in U$ we have

(4)
$$W(z) = Iw(z) = z + \sum_{j=2}^{\infty} \frac{1}{j} a_j z^j \ , \ z \in U$$

From remark 2.1 we have

(5)
$$R^n w(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j \ , \ n \in \mathbb{N} \ , \ z \in U$$

With simple calculations we obtain using (4) and (5):

(6)
$$(n+1)R^{n+1}w(z) - nR^n w(z) = z\left(R^n w(z)\right)' \ , \ n \in \mathbb{N}$$

and

(7)
$$z\left(R^n W(z)\right)' = R^n w(z) \ , \ n \in \mathbb{N} \ , \ z \in U$$

From here we obtain:

$$(n+1)R^{n+1}W(z) - nR^nW(z) = R^nw(z)\,,\; n \in \mathbb{N}\,,\; z \in U \tag{8}$$

or

$$(n+1)\frac{R^{n+1}W(z)}{R^nW(z)} - n = \frac{R^nw(z)}{R^nW(z)}\,,\; n \in \mathbb{N}\,,\; z \in U \tag{9}$$

With notations $\dfrac{R^{n+1}F(z)}{R^nG(z)} = p(z)$ and $\dfrac{R^{n+1}G(z)}{R^nG(z)} = h(z)$ we have:

$$z \cdot p'(z) = \frac{z \cdot (R^{n+1}F(z))'}{R^nG(z)} - \frac{R^{n+1}F(z)}{R^nG(z)} \cdot \frac{z \cdot (R^nG(z))'}{R^nG(z)}$$

and from (7) with $w(z) = f(z)$, $W(z) = F(z)$ and $w(z) = g(z)$, $W(z) = G(z)$ we have

$$z \cdot p'(z) = \frac{R^{n+1}f(z)}{R^nG(z)} - p(z) \cdot \frac{R^ng(z)}{R^nG(z)} = \frac{R^{n+1}f(z)}{R^ng(z)} \cdot \frac{R^ng(z)}{R^nG(z)} - p(z) \cdot \frac{R^ng(z)}{R^nG(z)} \tag{10}$$

From (9) with $w(z) = g(z)$ and $W(z) = G(z)$ we have

$$\frac{R^ng(z)}{R^nG(z)} = (n+1) \cdot h(z) - n \tag{11}$$

Now, from (10) and (11) we obtain

$$z \cdot p'(z) = [(n+1)h(z) - n] \cdot \left[\frac{R^{n+1}f(z)}{R^ng(z)} - p(z)\right]$$

or

$$\frac{R^{n+1}f(z)}{R^ng(z)} = p(z) + \frac{1}{(n+1)h(z) - n} \cdot zp'(z) \tag{12}$$

From remark 3.1 and remark 2.2 we have $\dfrac{R^{n+1}f(z)}{R^ng(z)} \prec Q_\delta(z)$ and $h(z) \prec Q_\delta(z)$, where $Q_\delta(z)$ is given by (3) and is convex with $Re\, Q_\delta(z) > \frac{1+\delta}{2}$. Using this results and the hypothesis we obtain $Re\,[(n+1)h(z) - n] > 0$, $z \in U$ and $p(z) + \dfrac{1}{(n+1)h(z) - n} \cdot zp'(z) \prec Q_\delta(z)$, with $Q_\delta(z)$ convex in U.

In this conditions from theorem 2.3 we have $\dfrac{R^{n+1}F(z)}{R^nG(z)} = p(z) \prec Q_\delta(z)$. Thus, from remark 3.1, we conclude that $F(z) = If(z) \in UQ_n(\delta)$ in respect to the function $G(z) = Ig(z) \in UK_n(\delta)$, with $n \in \mathbb{N}$, $\delta \in [-1, 1)$ and $\delta \geq \frac{n-1}{n+1}$.

Theorem 3.2 *If $f(z) \in UQ_n(\delta)$ in respect to the function $g(z) \in UK_n(\delta)$, with $n \in \mathbb{N}$, $\delta \in [-1, 1)$ and $c \in \mathbb{C}$ wtih $Re\, c \geq \frac{n(1-\delta)-(1+\delta)}{2}$, then $F(z) = I_cf(z) \in UQ_n(\delta)$ in respect to the function $G(z) = I_cg(z) \in UK_n(\delta)$, where I_c is the Bernardi integral operator defined by (2)*

Proof.

From theorem 2.1 we have $G(z) = I_c g(z) \in UK_n(\delta)$ in the conditions from the hypothesis.

If we consider $w(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $z \in U$ we have

$$W(z) = I_c w(z) = z + \sum_{j=2}^{\infty} \frac{c+1}{c+j} a_j z^j \, , \; z \in U \tag{13}$$

In a similar way with the proof of the theorem 3.1 (see relations (5),(6)) we obtain:

$$(c+1)R^n w(z) - cR^n W(z) = z\left(R^n W(z)\right)' \, , \; n \in \mathbb{N} \tag{14}$$

and

$$(c+1)R^n w(z) = (n+1)R^{n+1} W(z) + (c-n)R^n W(z) \, , \; n \in \mathbb{N} \tag{15}$$

From (15) we obtain

$$(c+1)\frac{R^n w(z)}{R^n W(z)} = (n+1)\frac{R^{n+1} W(z)}{R^n W(z)} + (c-n) \, , \; n \in \mathbb{N} \, , \; z \in U \tag{16}$$

With notations $\dfrac{R^{n+1}F(z)}{R^n G(z)} = p(z)$ and $\dfrac{R^{n+1}G(z)}{R^n G(z)} = h(z)$ by a similar calculus as the above theorem we have:

$$\frac{R^{n+1} f(z)}{R^n g(z)} = p(z) + \frac{1}{(n+1)h(z) + (c-n)} \cdot z p'(z) \tag{17}$$

From remark 3.1 and remark 2.2 we have $\dfrac{R^{n+1}f(z)}{R^n g(z)} \prec Q_\delta(z)$ and $h(z) \prec Q_\delta(z)$, where $Q_\delta(z)$ is given by (3) and is convex with $Re\, Q_\delta(z) > \frac{1+\delta}{2}$. Using this results and the hypothesis we obtain $Re\left[(n+1)h(z) + c - n\right] > 0$, $z \in U$ and

$$p(z) + \frac{1}{(n+1)h(z) + (c-n)} z p'(z) \prec Q_\delta(z)$$

, with $Q_\delta(z)$ convex in U.

In this conditions from theorem 2.3 we have $\dfrac{R^{n+1}F(z)}{R^n G(z)} = p(z) \prec Q_\delta(z)$, and thus the proof is complete.

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