BOUNDS OF THE COEFFICIENTS FOR UNIFORMLY CLOSE-TO-CONVEX FUNCTIONS

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Abstract. In this paper we find bounds of the coefficients of series expansion for so called n – uniformly close–to–convex of order α functions which had previously been defined as well as for the functions which can be obtained by applying a integral operator of Libera–Pascu type on the first ones. Thus there have been used the results given by N.N.Pascu, S.Kanas, T.Yaguchi, A.Wisniowska and C.I.Magdaş.

1. INTRODUCTION. Denote by A the set of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in the open disk U,and by S its subclass consisting of all univalent functions. Subclasses of S functions have been lately studied which have geometrical properties such as uniformly convex UCV, uniformly starlike UST, (n, α) – uniformly convex etc. The coefficient problem and the relation ship among then is frequently studied.

In this paper we'll find bounds of coefficients of series expansion upon unit disk U for so called n – uniformly close-to-convex of type α functions previously introduced [2] as well as for the coefficients of the functions obtained from the latter by applying an integral operator of Libera–Pascu type.

PRELIMINARY RESULTS

Let be the differential operator $D^n f$ defined by G.Sălăgean recurrently; $D^n : A \to A$, $n \in N \cup \{0\}$

$$\begin{aligned} D^0 f(z) &= f(z) \\ D' f(z) &= Df(z) = zf'(z) \\ D^n f(z) &= D(D^{n-1} f(z)) \; z \in U. \end{aligned} \tag{1}$$

1991 *Mathematics Subject Classification.* 30C45.
Key Words and Phrases: (n, α) - uniformly convex, n - uniformly close-to-convex of type α, integral operator of Libera–Pascu type.

DEFINITION 1. [7]. *Let $f \in S$, $\alpha \geq 0$, $n \in N \cup \{0\}$. We say that f is (α, n)–uniformly convex if*

$$(2) \qquad Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) > \left|\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right| \quad z \in U.$$

We denote by $(\alpha, n) - UCV$ the set of this functions.

Independently C.I.Magdaş [4] give a similar definition but named this functions $n-$ uniformly starlike of type α and denote by $USn(\alpha)$ the set of this functions.

For cases $n = 0$ and $n = 1$ on obtain the classes $\alpha - ST$ respectively $\alpha - UCV$ defined and detailed studied by S.Kanas and A.Wisniowska [5], [6].

Observe that the family of (n, α)–uniformly convex functions describes the class of analytic functions f, for that the expression $D^{n+1}f(z)/D^n f(z)$, $z \in U$, $n \in N \cup \{0\}$ lies in the conic region Ω_α which depends on the parameter α.

Denote by $\mathcal{P}(p_\alpha)$ [5], [6] $(0 \leq \alpha < \infty)$, the family of the functions p, such that $p \in \mathcal{P}$, and $p \prec p_\alpha$ in U, where the function p_α, maps the unit disk conformally onto the region Ω_α, such that $1 \in \Omega_\alpha$ and

$$(3) \qquad \partial\Omega_\alpha = \{u + iv : u^2 = \alpha^2(u-1)^2 + \alpha^2 v^2\}$$

The domain Ω_α is elliptic for $\alpha > 1$, hyperbolic, when $0 < \alpha < 1$ parabolic, when $\alpha = 1$, and a right half–plane when $\alpha = 0$ (for complete information see [5]).

Here $\mathcal{P}$ denotes the well known class of Caratheodory functions. The functions, which play the role of extremal functions of the classes $\mathcal{P}(p_\alpha)$ have been obtained in [5].

Obviously

$$p_0(z) = \frac{1+z}{1-z} \text{ and } p_1(z) = 1 + \frac{2}{\Pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad z \in U$$

(see [8]) and when $0 < \alpha < 1$, [5]

$$p_\alpha(z) = \frac{1}{1-\alpha^2}\cos\left\{Ai\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{\alpha^2}{1-\alpha^2}, \quad z \in U$$

or equivalently

$$p_\alpha(z) = \frac{1}{2(1-\alpha^2)}\left[\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)^A + \left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^A\right] - \frac{\alpha^2}{1-\alpha^2}$$

where $A = \dfrac{2}{\pi}\arccos\alpha$. Finally when $\alpha > 1$, the function p_α has the form (cf.[5], [6])

$$p_\alpha(z) = \frac{1}{\alpha^2-1}\sin\left(\frac{\pi}{2K(k)}\int_0^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right) + \frac{\alpha^2}{\alpha^2-1}$$

with

$$u(z) = [z - \sqrt{k}] \diagup [1 - \sqrt{k} z], \quad 0 < k < 1 \quad z \in U$$

and k is chosen such that $k = \cos h[\pi K'(k)] \diagup [4K(k)]$. $K(k)$ is Legendre's complete elliptic integral of the first kind, and $K'(k)$ is complementary integral of $K(k)$. Denote the coefficients of p_α by q_n. Then we have

$$p_\alpha(z) = 1 + q_1 z + q_2 z^2 + \ldots \quad z \in U$$

Remark 1. The domain $p_\alpha(U) = \Omega_\alpha$, $\alpha \in [0, \infty)$, is convex then, as an immediate consequence of the well known Rogosinski result for subordinate functions we obtain for $p \in \mathcal{P}(p_\alpha)$ where $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, $z \in U$

$$(4) \qquad |p_n| \leq |q_1| := q_1(\alpha) = \begin{cases} \dfrac{8(\arccos \alpha)^2}{\pi^2(1-\alpha^2)} & 0 \leq \alpha < 1 \\ \dfrac{8}{\pi^2} & \alpha = 1 \\ \dfrac{\pi^2}{4\sqrt{k}(\alpha^2 - 1)K^2(k)(1+k)} & \alpha > 1 \end{cases}$$

Remark 2. [6]. Let the coefficient $q_1 := q_1(\alpha)$ be the function of the variable α. Then, the function $q_1(\alpha)$ as regards α is positive, and strictly decreasing on the interval $[0, \infty]$ and it values are included in the interval $(0, 2]$ with $q_1(0) = 2$, $q_1(\sqrt{2} \diagup 2) = 1$, $q_1(1) = \dfrac{8}{\pi^2} = 0,81$. The values of $q_1(\alpha)$ tend to 0 when $\alpha \to \infty$ and so p_n as regards the parameter α.

DEFINITION 2. [2] *Let be $f \in A$, $\alpha \geq 0$, $n \in N \cup \{0\}$. We say that f is n–uniformly close-to-convex of α type if exist a function $g \in (\alpha, n) - UCV$ (or $US_n(\alpha)$) so that*

$$Re\left(\frac{D^{n+1}f(z)}{D^n g(z)}\right) > \alpha \left|\frac{D^{n+1}f(z)}{D^n g(z)} - 1\right|, \quad z \in U$$

We note the set of this functions by $UCC_n(\alpha)$.

Remark 3. $f \in UCC_n(\alpha)$ if and only if $h(z) = D^{n+1}f(z) \diagup D^n g(z)$ takes all values in Ω_α.

We note by La the integral operator $La : A \to A$ as

$$(5) \qquad f(z) = La(F(z)) = \frac{1+a}{z^a} \int_0^z F(t) t^{a-1} dt$$

when $a \in C$, $Re\ a \geq 0$, introduced by Libera for $a = 1$ then later generalized for the largest domain of parameter a ($a \in C, Re\, a \geq 0$) by N.N.Pascu.

THEOREM A. [1] *For every* $a \in C$, $Re\, a \geq 0$, $\alpha \geq 0$, $n \in N \cup \{0\}$

$$La[UCC_n(\alpha)] \subset UCC_n(\alpha)$$

On the other worlds by applying the operator La *on obtain a subclass of* $UCC_n(\alpha)$. *We denote by* $UCCP_n(\alpha)$ *this set.*

MAIN RESULTS

In the following we give bounds for the coefficients of series expansion for this functions belonging of the sets $UCC_n(\alpha)$ and $UCCP_n(\alpha)$.

THEOREM 1. *If* $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$ *belong to the class* $UCC_n(\alpha)$ *then*

$$|a_j| \leq \frac{\prod_{K=0}^{j-2} (q_1 + K)}{j^n (j-1)!} \tag{6}$$

where $q_1 = q_1(\alpha)$ *is given by (4) from Remark 1.*

Proof. Let $f \in UCC_n(\alpha)$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, and $g \in (\alpha, u) - UCV$,

$g(z) = z + \sum_{j=2}^{\infty} b_j z^j$.

According to the results of [7] we have the estimation

$$|b_j| \leq \frac{q_1(q_1+1)\dots(q_1+j-2)}{(j-1)!j^n} \quad j \geq 2 \tag{7}$$

If $f \in UCC_n(\alpha) \Longleftrightarrow h(z) = D^{n+1}f(z) / D^n g(z) \prec p_\alpha(z)$ where $p_\alpha(U) = \Omega_\alpha$.

Let $h(z) = 1 + c_1 z + c_2 z^2 + \dots$, then using the consequences of the subordination relationships as well as Rogosinski theorem, from (4) we get

$$|c_j| \leq q_1(\alpha) \tag{8}$$

Easy to observe that

$$D^{n+1} f(z) = z + \sum_{j=2}^{\infty} j^{n+1} c_j z^j$$

and

$$D^n g(z) = z + \sum_{j=2}^{\infty} j^n b_j z^j$$

Then by identifying the equal powers, we get

$$j^{n+1} a_j = c_j + 2^n b_2 c_{j-2} + 3^n b_3 c_{j-3} + \dots + c_1 (j-1)^n b_{j-1} + j^n b_j.$$

By using inequalities from (7) and (8) it follows:

$$j^{n+1}|a_j| \le q_1 \left[1 + 2^n|b_2| + 3^n|b_3| + \cdots + (j-1)^n|b_{j-1}| \right] + j^n|b_j|$$

or

$$j^{n+1}|a_j| \le q_1 \left[1 + \sum_{l=2}^{j-1} \frac{\prod_{K=0}^{l-2}(q_1+K)}{(l-1)!} \right] + \frac{\prod_{K=0}^{j-2}(q_1+K)}{(j-1)!}$$

by mathematical induction we find that:

$$1 + \sum_{l=2}^{j-1} \frac{\prod_{K=0}^{l-2}(q_1+K)}{(l-1)!} = \frac{\prod_{K=1}^{j-2}(q_1+K)}{(j-2)!}$$

Thus

$$j^{n+1}|a_j| \le q_1 \frac{\prod_{K=1}^{j-2}(q_1+K)}{(j-2)!} + \frac{\prod_{K=0}^{j-2}(q_1+K)}{(j-1)!} =$$

$$= \frac{\prod_{K=1}^{j-2}(q_1+K)}{(j-1)!}[q_1(j-1)+q_1] = \frac{\prod_{K=1}^{j-2}(q_1+K)}{(j-1)!} q_1 j$$

or

$$|a_j| \le \frac{\prod_{K=0}^{j-2}(q_1+K)}{j^n(j-1)!}.$$

For special values of the parameters n, α on obtain bounds of the coefficients given by F.Ronning, W.Ma, D.Minda respectively S.Kanas ([4], [5], [6], [8]).

THEOREM 2. *Let $a \in C$, $Re\ a \ge 0$, $n \in N \cup \{0\}$, $\alpha \ge 0$. If $F \in UCC_n(\alpha)$, $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$ then for $f(z) = La\ F(z)$, with $f(z) = z + \sum_{j=2}^{\infty} b_j z^j$, where La is the Libera–Pascu integral operator (5) we get*

$$|b_j| \le \left| \frac{a+1}{a+j} \right| \frac{\prod_{K=0}^{j-2}(q_1+K)}{j^n(j-1)!} \quad j = 2, 3, \ldots$$

where $q_1 = q_1(\alpha)$ is given by (4).

Proof. We observe that $f(z) = La\ F(z)$ is equivalent with

(9) $$(1+a)F(z) = a\ f(z) + z\ f'(z) \quad a \in C, Re\ a \ge 0$$

If in (9) we put f and F with the above forms then by identifying the coefficients of the terms in z^j we get

$$b_j \cdot (a+j) = (1+a)a_j$$

According to (6) it follows that

$$|b_j| \leq \left|\frac{a+1}{a+j}\right| \frac{\prod_{K=0}^{j-2}(q_1+K)}{j^n(j-1)!}$$

which complete the proof of theorem 2.

For $a = 1$ we have

$$|a_j| \leq \frac{2\prod_{K=0}^{j-2}(q_1+K)}{j^{n-1}(j+1)!}.$$

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